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"A SUDY OF FIXED POINT THEORY IN BANACH SPACES"

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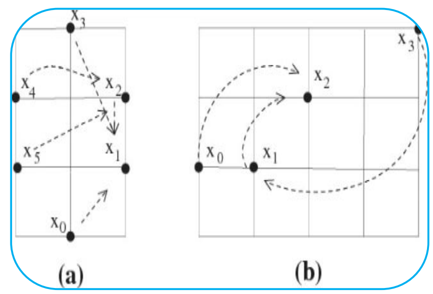
ABSTRACT

The theory of Banach spaces developed in parallel with the general theory of linear topological spaces. These theories mutually enriched one another with new ideas and facts. Thus, the idea of semi-norms, taken from the theory of normed spaces, became an indispensable tool in constructing the theory of locally convex linear topological spaces. The ideas of weak convergence of elements and linear functionals in Banach spaces ultimately evolved to the concept of weak topology. The theory of Banach spaces is a thoroughly studied branch of functional analysis, with numerous applications in various branches of mathematics - directly or by way of the theory of operators.



KEYWORDS : *Banach Spaces, Theory, Functional Analysis and Fixed Point.*

INTRODUCTION

The function spaces introduced by D. Hilbert, M. Fréchet and F. Riesz between 1912 and 1918 served as the starting point for the theory of Banach spaces. It is in these spaces that the fundamental concepts of strong and weak convergence, compactness, linear functional, linear operator, etc., were originally studied. Banach spaces were named after S. Banach who in 1922 began a systematic study of these spaces, based on axioms introduced by himself, and who obtained highly advanced results.

Fixed point theory is a developing branch of mathematics related with function analysis and topology. These theorems are used in proving the existence of Differential equation Partial differential equation, Random differential equation, and integral equation in other related field. They are used to study the Eigen value problem, boundary value problem as well as approximation theory, variational inequality and complementary problems. It is also very fruitful in non linear functional analysis and in approximation theory of optimization .Fixed point theory is considered as a powerful tool in demonstrating the existence of solution of various equations of various fields of applied mathematics.

There are many types of problems that have been determined through the use of fixed point theorems include periodic points, equilibrium points, extreme points, optical points and stability point. The different techniques that have been used for the determination of fixed point are based upon. Banach contraction principle, non-expansive mapping, and retraction and deformation techniques. The fixed point theory can be classified into three major types on the basis of the varieties of problems.

DISCUSSION:

First method is based on general topology and functional analysis. It includes the works Poincare (1899), Brouwer (1910), Banach (1924). Schauder (1927), Browder (1960) and various other authors, The second method used to determined the existence of fixed point exploit algebraic tools on based on Homotopy and Homology. These techniques are more accurate and efficient than the classical methods of first types. Also they provide solution of various problems which cannot be determined by the classical method of topology. These methods consists the works of Nielsen (1921), Lefschetz-Hopf (1935), Leray-Schauder (1945), Schirmer (1967), Brown (1990), as well as many others in this field.

Fixed point theory is a rich, interesting and exciting branch of mathematics. It is a relatively young but fully developed area of research. Study of the existence of fixed point's falls within several domains such as classical analysis, functional analysis, and operator theory, general and algebraic topology. Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as topology, mathematical economics, game theory, and approximation theory and initial and boundary value problems in ordinary and partial differential equations.

Moreover, recently, the usefulness of this concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major tool in the arsenal of mathematics.

Fixed point theory is equivalent to best approximation, variation inequality and the maximal elements in mathematical economics. The sequence of iterates of fixed point theory can be applied to find a solution of a variation inequality and the best approximation theory. The theory of fixed points is concerned with the conditions which guarantee that a map $T: X \rightarrow X$ of a topological space X into itself. Admits one or more fixed points that are points x in X for which $x = Tx$. For example, a translation, i.e. the mapping $T(x) = x + a$ for a fixed a, has no fixed point, a rotation of the plane has a single fixed point (the center of rotation), the mapping $x \rightarrow x^2$ of R into itself has two fixed points (0 and 1) and the projection $(\xi_1, \xi_2) \rightarrow \xi_1$ of R^2 into the ξ_1 – axis has infinitely many fixed points (all points of the ξ_1-axis). Existence problems of the type $(T - I)x = 0$ arise frequently in the analysis. For example, the problem of solving the equation $p(z) = 0$, where p is a complex polynomial, is equivalent to find a fixed point of the self maps z–p(z) of C . More generally, it $D : M \rightarrow E$ is an operator acting on a subset M of a linear space E, to show that the equation $Du = 0$ [resp.$u - \lambda Du=0$] has a solution, is equivalent to show that the map $y \rightarrow y - Dy$ [resp. $y \rightarrow \lambda Dy$] has a fixed point . The earliest fixed point theorem is that of Brouwer, who in (1912), proved that a continuous self-mapping T of the closed unit ball R^n has at least one fixed point, that is, a point x such that $Tx = x$. Several proof of this historic result can be found in the existing literature. Another fundamental result after Brouwer's fixed point theorem was given by polish mathematicians S. Banach in (1922). Banach proved a theorem, which ensures under appropriate conditions, the existence and uniqueness of a fixed point. This result is popularly known as "Banach fixed point theorems" or the "Banach Contraction Principle". It states that a contraction mapping of a complete metric space into itself has a unique fixed point. It is the simplest and one of the most versatile results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map, it produces approximations of any required accuracy. Due to its applications in various disciplines of mathematics and mathematical sciences, the Banach contraction principle has been extensively studied and generalized on many settings and fixed point theorems have been established.

There are large classes of mappings for which fixed point theorems have been studied. It includes contractive mappings, contraction of various order mappings, Ciric contraction, asymptotically regular, densifying etc. Apart from single mappings, pair of mappings, the sequence of mappings and family of mappings are also some of the classes of mappings that have interested mathematicians.

Along with the theoretical aspects of fixed point problem, the fixed point algorithms were the major breakthrough in computational method, which five us for first time a numerical algorithm for approximation. The first work was proposed by Noble Laureates H. E. Scarf in 1967 and since then many algorithms have been proposed having as a basis different proofs of Brouwer's theorem. These different types of algorithm were proposed by Freudential, Kuhn, Merill, Eaves, Saigal etc.

$d(x,x) = \ , \forall x \in 0.$

It should be noted that every metric is a pseudo- metric but a pseudo-metric is not necessarily a metric.

A fixed point is a point x such that $f(x) = x$ Graphically, these are exactly those points where the graph of f, whose equation is $y = f(x)$, crosses the diagonal, whose equation is $y = x$. You can often solve for them exactly:

If $| f'(x) | < 1$ and x_{n-1} is close to x, then x_n will be even closer to x (by a factor of $| f'(x) |$ approximately) and hence as n gets larger, we will get convergence to x.

We can now introduce the official names. A fixed point x is called attracting if starting with some number sufficiently close to x and iterating it always leads to convergence to x. Our conclusion is that:

If a fixed point has $| f'(x) | < 1$, it is attracting on the other hand, a fixed point that pushes away nearby values is called repelling. One can show by a similar analysis that:

If a fixed point has $| f'(x) | > 1$, it is repelling.

The borderline situation $f'(x) = \pm 1$ needs to be looked at on a case-by-case basis (may be attracting, repelling, or show mixed behaviour). Now we can review our previous examples in this light:

Examples: $f(x) = \frac{x}{2} + \frac{1}{x}$ has a fixed point at $x = \sqrt{2}$, and $f'(\sqrt{2}) = 0$. So this is very very attracting. $f(x) = 5x/2 - 3x^2/2$ has a fixed point at $x = 1$, and $f'(1) = 5/2 - 1 = -1/2$, so it is attracting (the negative sign means that we reach the fixed point by bouncing around from left to right). By the way, there's another fixed point at $x = 0$, but there $f'(0) = 5/2$, so that one is repelling.

$f(x) = 13x/4 - 3x^2/2$ has a fixed point at $x = 3/2$, and $f'(3/2) = 13/4 - 9/2 = -5/4$, so it is repelling.

As we can see from the last two examples, changing a parameter (in this case a) can have the effect of changing the situation from attracting to repelling. Some of the phenomena you hear referred to as "tipping points" are of this kind.

Fixed point theorems concern maps f of a set X in to it self that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology.

Theorem:

Assume that each f_n has at least a fixed point $x_n = f_n(x_n)$. Let $f : X \to X$ be a uniformly continuous map such that f^m is a contraction for some $m \geq 1$. If f_n converges uniformly to f, then x_n converges to $\bar{x} = f(\bar{x})$.

Proof: We first assume that f is a contraction (i.e., m = 1). Let $\lambda < 1$ be the Lipschitz constant of f. Given $\varepsilon > 0$, choose $n_0 = n_0(\varepsilon)$ such that

$$d(f_n(x), f(x)) \leq \varepsilon (1 - \lambda), \ \forall n \geq n_0, \forall x \in X.$$

Then, for $n \geq n_0$,

$$\begin{aligned} d(x_n, \bar{x}) &= d(f_n(x_n), f(\bar{x})) \\ &\leq d(f_n(x_n), f(x_n)) + d(f(x_n), f(\bar{x})) \\ &\leq \varepsilon (1 - \lambda) + \lambda\, d(x_n, \bar{x}). \end{aligned}$$

Therefore $d(x_n, \bar{x}) \leq \varepsilon$, which proves the convergence.
To prove the general case it is enough to observe that if

$$d(f^m(x), f^m(y)) \leq \lambda^m d(x, y)$$

for some $\lambda < 1$, we can define a new metric d_0 on X equivalent to d by setting

$$d_0(x, y) = \sum_{k=0}^{m-1} d(f^k(x), f^k(y)) / \lambda^k.$$

Moreover, since f is uniformly continuous, f_n converges uniformly to f also with respect to d_0. Finally, f is a contraction with respect to d_0. Indeed,

$$d_0(f(x), f(y)) = \sum_{k=0}^{m-1} d(f^{k+1}(x), f^{k+1}(y)) / \lambda^k.$$

$$= \lambda \sum_{k=0}^{m-1} d(f^k(x), f^k(y)) / \lambda^k + d(f^m(x), f^m(y)) / \lambda^{m+1}$$

$$\leq \lambda \sum_{k=0}^{m-1} d(f^k(x), f^k(y)) / \lambda^k = \lambda d_0(x,y).$$

So the problem is reduced to the previous case m = 1.

Theorem is due to Nadler [Pacif. J. Math. (1968)] and Fraser and Nadler [Pacif. J. Math. (1969)]. Uniform convergence in Theorem cannot in general be replaced by point wise convergence. For instance, in every infinite-dimensional separable or reflexive Banach space there is a point wise convergent sequence of contractions whose sequence of fixed points has no convergent subsequence's.

CONCLUSION:

In the literature of fixed point theory, many authors have extensively studied. Common fixed point theorems in Banach spaces. In (1992) Aqueel Ahmad and Imdad obtained some common fixed point theorem for compatible asymptotic regular mappings in Banach spaces and generalized some known results with respect to their mappings and inequality conditions. Some non expansive mappings from a closed convex subset of uniformly convex Banach spaces into itself under some asymptotic contraction assumptions.

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