Indian Streams Research Journal
ISSN 2230-7850

Impact Factor : 3.1560(UIF)
Volume - 5 | Issue - 6 | July – 2015

THE COMPLEMENTARY DOMINATING ENERGY OF A GRAPH

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Abstract

For a graph G, let $D \subseteq V(G)$ be a dominating set of G. If $V - D$ contains a dominating set D^1 with respect to D, then D^1 is called a complementary (inverse) dominating set of G. The smallest cardinality among all complementary dominating sets of G is called the complementary domination number of G and it is denoted by $\gamma_c(G)$. In this paper, we study complementary dominating energy $E_{CD}(G)$ of a graph G. We are compute complementary dominating energies of some standard and well-known families of graphs. Upper and lower bounds for $E_{CD}(G)$ are established.

Mathematics Subject Classification: 05C50, 05C69.

Keywords: dominating set, complementary domination number, complementary dominating matrix, complementary dominating eigenvalues, complementary dominating energy.

Short Profile

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INTRODUCTION

In this paper, we consider a simple graph $G(V,E)$ without isolated vertex. We denoted by n and m to the number of its vertices and edges, respectively. We refer the reader to [9] for more graph theoretical analogist not defined here. A subset D of vertices set V of G is called a dominating set of G if every vertex $v \in V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. A dominating set D^1 contained in $V - D$ is called a complementary (an inverse) dominating set of G with respect to D. The smallest cardinality among all dominating sets in $V - D$ is called the complementary (inverse) domination number of G and it is denoted by $\gamma_c(G)$. Any complementary dominating set of G which has $\gamma_c(G)$ vertices is called a γ_c-set of G. If a graph G has no isolated vertices, then the complement $V - D$ of every minimal dominating set D contains a dominating set. Thus every graph without isolated vertices contains a complementary dominating set with respect to a minimum dominating set and so every graph has a complementary domination number. This concept of complementary domination was introduced by V. R. Kulli [15]. For more details in this concept see [4, 12, 16]. For more details in domination theory of graphs we refer to [10].

The concept of energy of a graph was introduced by I. Gutman [7] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G, i.e.

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

For more details on the mathematical aspects of the theory of graph energy see [2, 8, 18]. The basic properties including various upper and lower bounds for energy of a graph have been established in [17, 19], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [5, 6]. Recently C. Adiga et al [1] defined the minimum

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covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover *C*. Further, minimum dominating energy, Laplacian minimum dominating energy and minimum dominating distance energy of a graph *G* can be found in [11, 13, 14] and the references cited there in.

Motivated by these papers, we study complementary dominating energy $E_{CD}(G)$ of a graph *G*. We compute complementary dominating energies of some standard and well-known families of graphs. Upper and lower bounds for $E_{CD}(G)$ are established. It is possible that the upper dominating energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

The Complementary Domination Energy of a Graph

Let *G* be a graph of order *n* with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. Let *D* be a γ-set of a graph *G*. A dominating set $D^1 \subseteq V - D$ is called a complementary dominating set of *G* with respect to *D*. The complementary domination number $\gamma_c(G)$ of *G* is the cardinality of a smallest complementary dominating set of *G*. Any complementary dominating set D^1 with cardinality equals to $\gamma_c(G)$ is called minimum complementary dominating set of *G*. The complementary dominating matrix of *G* is the $n \times n$ matrix $A_{CD}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E; \\ 1, & \text{if } i = j \text{ and } v_i \in D'; \\ 0, & \text{othewise.} \end{cases}$$

The characteristic polynomial of $A_{CD}(G)$ is denoted by

$f_n(G,\lambda) := det(\lambda I - A_{CD}(G))$.

The complementary dominating eigenvalues of the graph *G* are the eigenvalues of $A_{CD}(G)$. Since $A_{CD}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The complementary dominating energy of *G* is defined as:

$$E_{CD}(G) = \sum_{i=1}^{n} |\lambda_i| .$$

We first compute the complementary dominating energy of a graph in Figure 1

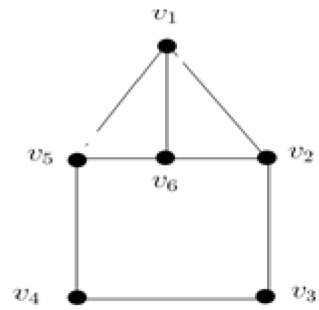

Figure 1

Let *G* be a graph in Figure 1, with vertices set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and let its dominating set be $D = \{v_1, v_3\}$. The complementary set of *G* with respect to *D* are

$$D'_1 = \{v_2, v_5\},\ D'_2 = \{v_4, v_6\} \text{ or } D'_3 = \{v_2, v_4\}$$

Then

$$A_{CD_1}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{CD1}(G)$ is

$$f_n(G,\lambda) = \lambda^6 - 2\lambda^5 - 7\lambda^4 + 7\lambda^3 + 13\lambda^2 - 4\lambda - 4.$$

Hence, the complementary dominating eigenvalues are $\lambda_1 \approx 3.4715$, $\lambda_2 \approx 1.6524$, $\lambda_3 \approx 1.3061$, $\lambda_4 \approx 0.6947$, $\lambda_5 \approx -0.4983$, $\lambda_6 \approx -1.6973$. Therefore the complementary dominating energy of *G* is

$$E_{CD1}(G) \approx 9.0035.$$

If we take another complementary dominating set of *G*, namely $D'_2 = \{v_4, v_6\}$, we get that

$$A_{CD_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of $A_{CD2}(G)$ is

$f_n(G,\lambda) = \lambda^6 - 2\lambda^5 - 7\lambda^4 + 6\lambda^3 + 14\lambda^2 - 4\lambda - 8.$

The complementary dominating eigenvalues are $\lambda_1 \approx 3.2361$, $\lambda_2 \approx 1.4142$, $\lambda_3 \approx 1.0000$ $\lambda_4 \approx -1$, $\lambda_5 \approx -1.2361$ $\lambda_6 \approx -1.4142$. Therefore the complementary dominating energy of G is

$E_{CD2}(G) \approx 9.3006.$

This example illustrates the fact that the complementary dominating energy of a graph G depends on the choice of the complementary dominating set. i.e. the complementary dominating energy is not a graph invariant.

In the following section, we introduce some properties of characteristic polynomials of complementary dominating matrix of a graph$G$.

Theorem 2.1. *Let G be a graph of order n, size m, complementary dominating set D and let*

$fn(G,\lambda) = c_0\lambda^n + c_1{}^{n-1} + c_2{}^{n-2} + ... + c_n$

be the characteristic polynomials of complementary dominating matrix of a graph G.

Then

1. $c_0 = 1$.
2. $c_1 = -|D|$
3. $$c_2 = \begin{pmatrix} |D| \\ 2 \end{pmatrix} - m$$

Proof. 1. From the definition of $f_n(G,\lambda)$.

2. Since the sum of diagonal elements of $A_{CD}(G)$ is equal to $|D|$. The sum of determinants of all 1 × 1 principal submatrices of $A_{CD}(G)$ is the trace of $A_{CD}(G)$, which evidently is equal to $|D|$. Thus, $(-1)^1 c_1 = |D|$.

3. $(-1)^2 c_2$ is equal to the sum of determinants of all 2 × 2 principal submatrices of $A_{CD}(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ij}^2 \\ &= \begin{pmatrix} |D| \\ 2 \end{pmatrix} - m. \end{aligned}$$

Theorem 2.2. *Let G be a graph of order n. Let $\lambda_1,\lambda_2,...,\lambda_n$ be the eigenvalues of $A_{CD}(G)$. Then*

$$(i)\ \sum_{i}^{n} \lambda_i = |D|.$$

$$(ii)\ \sum_{i}^{n} \lambda_i^2 = |D| + 2m.$$

Proof. (*i*) Since the sum of the eigenvalues of $A_{CD}(G)$ is the trace of $A_{CD}(G)$, then

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D|$$

(*ii*) Similarly the sum of squares eigenvalues of $A_{CD}(G)$ is the trace of $(A_{CD}(G))^2$. Then

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + \sum_{i\ne j}^{n} a_{ij}a_{ji} \\ &= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i<j}^{n} a_{ij}^2 \\ &= |D| + 2m. \end{aligned}$$

Bapat and S. Pati [3], proved that if the graph energy is a rational number then it is an even integer. Similar result for minimum dominating energy is given in the following theorem.

Theorem 2.3. *Let G be a graph with a complementary dominating set D. If the complementary dominating energy $E_{CD}(G)$ of G is a rational number, then*

$E_{CD}(G) \equiv |D| \pmod 2$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the complementary dominating eigenvalues of a graph G of which $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive and the rest are non-positive, then

$$\sum_{i=1}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \ldots + \lambda_n)$$

.

$= 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r) - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)$.

$= 2q - |D|$. *Where* $q = \lambda_1 + \lambda_2 + \ldots + \lambda_r$.

Since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are algebraic integers, so is their sum. Hence $(\lambda_1 + \lambda_2 + \ldots + \lambda_r)$ must be an integer if $E_{CD}(G)$ is rational. Hence the theorem.

The Complementary Dominating Energy of Some Graphs

In this section, we investigate the exact values of the complementary dominating energy of some standard and well-known graphs.

Theorem 3.1. *For $n \geq 2$, the complementary dominating energy of complete graph*

K_n is

$$E_{CD}(K_n) = (n-2) + \sqrt{(n^2 - 2n - 5)}.$$

Proof. For complete graphs K_n, let $D^1 = \{v_i\}$ for $1 \leq i \leq n$ be the complementary dominating set with respect the dominating set $D = \{v_j\}$ for $i \neq j$, $1 \leq j \leq n$. Since, the complementary dominating number is equal to the domination number (namely one). Hence, we get the complementary dominating matrix A_{CD} from the minimum dominating matrix A_D [14] by pair of rearranging the rows and columns.

Therefore,

$$E_{CD}(K_n) = E_D(K_n) = (n-2) + \sqrt{(n^2 - 2n - 5)}$$

Theorem 3.2. *For the complete bipartite graph $K_{r,r}$, $2 \leq r$, the complementary dominating energy is equal to*

$$(r+1) + \sqrt{r^2 + 2r - 3}.$$

Proof. For the complete bipartite graph $K_{r,r}$, ($2 \leq r$) with vertex set $V = (V_1, V_2)$ where V_1 and V_2 are the partite sets of its, $V_1 = \{v_1, v_2, \cdots, v_r\}$ and $V_2 = \{u_1, u_2, \cdots, u_r\}$. The dominating set is $D = \{v_1, u_1\}$ and the complementary dominating set with respect to D is $D^1 = \{v_2, u_2\}$. Hence, the complementary dominating matrix of $K_{r,r}$
get it from $A_D(K_{r,r})$ by Pair of rearranging the rows and columns. Then

$$A_{CD}(K_{r,r}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2r \times 2r}$$

The characteristic polynomial of $A_{CD}(K_{r,r})$ is

$$f_n(K_{r,r}, \lambda) = \begin{vmatrix} \lambda - 1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \lambda - 1 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2r \times 2r}$$

$$= \lambda^{2r-4} \left(\lambda^2 + (r-1)\lambda - (r-1)\right) \left(\lambda^2 - (r+1)\lambda - (r-1)\right)$$

The spectrum of $K_{r,r}$ is *CD Spec*$(K_{r,r})$ =

$$\begin{pmatrix} 0 & \frac{-(r-1)+\sqrt{r^2+2r-3}}{2} & \frac{-(r-1)-\sqrt{r^2+2r-3}}{2} & \frac{(r+1)+\sqrt{r^2-2r+5}}{2} & \frac{(r+1)-\sqrt{r^2-2r+5}}{2} \\ 2r-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the complementary dominating energy of a complete bipartite graph is

$$E_{CD}(K_{r,r}) = (r+1) + \sqrt{r^2 + 2r - 3}.$$

Theorem 3.3. *For $n \geq 2$, the complementary dominating energy of a star graph $K_{1,n-1}$ is equal to*

$$(n-2) + \sqrt{4n - 3}$$

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, where v_0 is the central vertex. Since the minimum dominating set of $K_{1,n-1}$ is $D = \{v_0\}$ it follows that the complementary dominating set of $K_{1,n-1}$ with respect to D is $D^1 = \{v_1, v_2, \ldots, v_{n-1}\}$. Hence, the complementary dominating matrix of $K_{1,n-1}$ is

$$A_{CD}(K_{1,n-1}) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}$$

The characteristic polynomial of $A_{CD}(K_{1,n-1})$ is

$$f_n(K_{1,n-1}, \lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & 0 & \cdots & 0 \\ -1 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}$$

$$= (\lambda - 1)^{n-2} \left[\lambda^2 - \lambda - (n-1)\right].$$

The spectrum of $K_{1,n-1}$ is

$$CD\ Spec(K_{1,n-1}) = \begin{pmatrix} 1 & \frac{1+\sqrt{1+4(n-1)}}{2} & \frac{1-\sqrt{1+4(n-1)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}$$

Therefore, the complementary dominating energy of a star graph is

$$E_{CD}(K_{1,n-1}) = (n-2) + \sqrt{4n-3}.$$

Definition 3.4. *The double star graph $S_{n,m}$ is the graph constructed from union $K_{1,n-1}$ and $K_{1,m-1}$ by join whose centers vertices v_0 and u_0 by an edge. A vertex set*

$V(S_{n,m}) = \{v_0, v_1, \ldots, v_{n-1}, u_0, u_1, \ldots, u_{m-1}\}$ and edge set $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j : 1 \le i \le n-1,\ 1 \le j \le m-1\}$. Therefore, double star graph is bipartite graph.

Theorem 3.5. *For $m \ge 3$, the upper dominating energy of double star graph $S_{m,m}$ is equal to*

$$(2m - 4) + 2\sqrt{m} + 2\sqrt{m-1}$$

Proof. For the double star graph $S_{m,m}$ with vertex set $V = \{v_0, v_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{m-1}\}$ the dominating set is $D = \{v_0, u_0\}$. Hence the complementary dominating set of $S_{m,m}$ withe respect to D is $D^1 = \{v_1, v_2, \ldots, v_{m-1}, u_1, u_2, \ldots, u_{m-1}\}$. Then

$$A_{CD}(S_{m,m}) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 1 \end{pmatrix}_{2m \times 2m}$$

The characteristic polynomial of $A_{CD}(S_{m,m})$ is

$$f_m(S_{m,m}, \lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda-1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & \lambda-1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda-1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda & -1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & \lambda-1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}$$

$$= (\lambda - 1)^{2m-4} \left(\lambda^2 - m\right) \left(\lambda^2 - 2\lambda - (m-2)\right)$$

Hence,

$$CD\ Spec(S_{m,m}) = \begin{pmatrix} 1 & \sqrt{m} & -\sqrt{m} & 1+\sqrt{m-1} & 1-\sqrt{m-1} \\ 2m-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the complementary dominating energy of double star graph is

$$E_{CD}(S_{m,m}) = (2m - 4) + 2\sqrt{m} + \sqrt{m-1}$$

Definition 3.6. *The cocktail party graph, denoted by $K_{2\times p}$, is a graph having vertex*

set $V(K_{2\times p}) = \bigcup_{i=1}^{p} \{ui, vi\}$ and edge set $E(K_{2\times p}) = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \le i < i=1\ j \le p\}$. i.e. $n = 2p$, $m = \frac{p^2-3p}{2}$ and for ever v ⍰ $V(K_{2\times p})$, $d(v) = 2p - 2$.

Theorem 3.7. *For the cocktail party graph $K_{2\times p}$ of order $n = 2p$, $p \ge 3$, the complementary dominating energy is equal to*

$$(2p-3) + \sqrt{4p^2 - 4p - 9}.$$

Proof. For cocktail party graphs $K_{2\times n}$ with minimum domination set $D = \{u_i, v_i\}$ for $1 \le i \le p$ the complementary dominating set with respect to D is $D^1 = \{u_j, v_j\}$ for $j \ne i$ and $1 \le j \le p$. Hence, for cocktail party graphs the complementary dominating matrix get from a minimum dominating matrix [14] by rearranging the rows. Therefore the complementary dominating energy

$$E_{CD}(K_{2\times n}) = E_D(K_n) = (2p-3) + \sqrt{4p^2-4p-9}.$$

4Bounds for Complementary Domination Energy of a Graph

In this section we shall investigate with some bounds for complementary dominating energy of a graph.

Theorem 4.1. *Let G be a graph of order n and size m. Then*

$$\sqrt{2m + \gamma c(G)} \leq E_{CD}(G) \leq \sqrt{n(2m + \gamma c(G))}$$

Proof. Consider the Couchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choose $a_i = 1$ and $b_i = |\lambda_i|$, we get

$$(E_{CD}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \lambda_i^2\right) \leq n(2m+|D|) \leq n(2m+\gamma_c(G)).$$

Therefore, the upper bound is hold. For the lower bound, since

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \geq \sum_{i=1}^{n} \lambda_i^2$$

Then

$$(E_{CD}(G))^2 \geq \sum_{i=1}^{n} \lambda_i^2 = 2m + |D| = 2m + \gamma_c(G).$$

Therefore,

$$E_{CD}(G) \geq \sqrt{2m+\gamma_c(G)}.$$

Similar to McClellands [19] bounds for energy of a graph, bounds for $E_{CD}(G)$ are given in the following theorem.

Theorem 4.2. *Let G be a graph of order and size n and m, respectively. If P = det(A_{CD}(G)), then*

$$E_{CD}(G) \geq \sqrt{2m + \gamma_c(G) + n(n-1)P^{2/n}}.$$

Proof. Since

$$(E_{CD}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)\left(\sum_{i=1}^{n} |\lambda_i|\right) = \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{i \neq j} |\lambda_i||\lambda_j|$$

Employing the inequality between the arithmetic and geometric means, we get

$$\frac{1}{n(n-1)} \sum_{i\neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}.$$

Thus

$$(E_{UD}(G))^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}$$
$$\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=j} |\lambda_i|^{2(n-1)}\right)^{1/[n(n-1)]}$$
$$= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left|\prod_{i\neq j} \lambda_i\right|^{2/n}$$
$$= 2m + \gamma_c(G) + n(n-1)P^{2/n}.$$

This completes the proof.∎

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