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# A NEW ITERATIVE METHOD OF ESTIMATION FOR NONLINEAR REGRESSION MODEL

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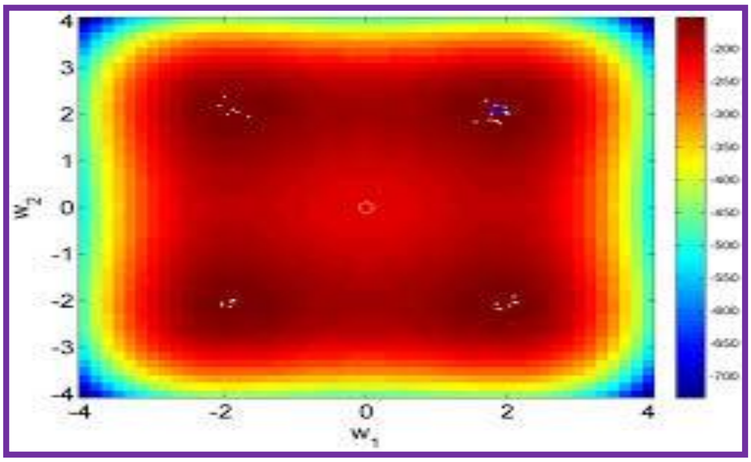
**ABSTRACT :**

Nonlinear regression analysis is currently the most fertile area of research in the modern theory of mathematical science. It is a powerful technique for analyzing data described by models which are nonlinear in parameters. These models are two types namely (i) Nonlinear regression models that are intrinsically linear and (ii) Nonlinear regression models that are intrinsically nonlinear.

In the case of intrinsically linear models, the ordinary least squares (OLS) estimation can be applied to the transformed models and the optimal estimators can be obtained for the parameters. In the case of intrinsically nonlinear regression models, OLS estimation fails to give optimal estimators. In this case, the OLS estimation can be applied under Iterative procedure for estimating parameters of the nonlinear models.

This research paper gives a new iterative method to estimate the parameters of the nonlinear regression model that is intrinsically nonlinear.

**KEYWORDS:** Nonlinear Regression Model; Nonlinear Least Squares (NLLS) estimation; Iterative process.




## I. INTRODUCTION

A model may be considered as a mathematical description of a physical, chemical or biological state or process. Many models used in Applied Mathematics and Mathematical Statistics are nonlinear in nature. One of the major topics in the literature of Theoretical and Applied Mathematics is the estimation of parameters of nonlinear regression models. A perfect model may have too many parameters to be useful.

A great deal of research in mathematical modelling has been directed to the nonlinear modelling and establishing functional relationships among different variables. Nonlinear models have a wide number of applications in physical, biological and social sciences, business, economics, engineering and management sciences. Now-a-days, nonlinear model building is new and very fascinating filed of research in Applied mathematical sciences.

A large number of problems in the nonlinear model building are concerned with the inferential aspects including estimating the parameters and testing the hypothesis about the parameters of the nonlinear regression models. Now-a-days, efficient estimation of the nonlinear regression models has received little attention. The literature on nonlinear methods of estimation has been grown enormously for the past four decades.

The main contributions in the field of nonlinear regression models have been made by Gallant, Rossi and Tauchen (1933), Levenberg (1944), Hartley (1961), Jenrich (1969), Goldfeld and Quandt (1972), Biggs (1971), Ross (1971), Chambers (1973), Gallant (1975 a, 1975 b), Bates and Watts (1980, 2008), Dennis, Gay and Welsch (1981). Hiebert (1981), McCullagh (1983), Ratkowsky (1983), Dennis and Schnabel (1983), Cordeiro and Paula (1989), Taylor and Uhlig (1990), Cameron and Windmeijer (1997), Ord, Koehler and Snyder (1997), Davidson and Mac Kinnon (1999), Popli (2000), Fox (2002), Smyth (2002), Davidian and Giltinan (2003), Vasilyev (2008), Fox and Wiesberg (2010), Potocky and Stehlik (2010), Grafarend and Awange (2012) and others.

In the present study, an attempt has been made by developing a new iterative technique for estimating parameters of nonlinear regression model.

## II. SPECIFICATION OF NONLINEAR REGRESSION MODEL:

Consider the standard nonlinear regression model

$$Y_i = f\left(X_i, \beta\right) + \varepsilon_i, \qquad i = 1, 2, ..n$$

Where $X_i = \left(X_{i1}, X_{i2}, ....X_{ik}\right)$ is a k-component vector denoting the $i^{th}$ observation on known k-explanatory variables;

$\beta$ is a $p \times 1$ vector of unknown parameters; $f\left(.\right)$ is a known twice differentiable function;

$\varepsilon_i, i = 1, 2, ..., n$ are independent identically normally distributed random variables with mean zero and unknown constant variance $\sigma^2$ ; and $\varepsilon_i$ and $X_i$ are independent random variables.

The usual assumptions of the nonlinear regression model are given by:

(i) The conditional mean function for $Y_i$ given $X_i$ is $f\left(X_i, \beta\right)$ i.e.

$$E\left[Y_i / X_i\right] = f\left(X_i, \beta\right), \qquad i = 1, 2, ..., n$$

Here, $f\left(X_i, \beta\right)$ is a continuously differentiable function of .

(ii) The parametric vector $\beta$ is estimable or identified.

(iii) The error observation $\varepsilon_i$ is uncorrelated with the conditional mean function for all observations in the sample.

i.e., $E\left[\varepsilon_i / f\left(X_i, \beta\right)\right] = 0$

(iv) $\varepsilon_i$'s are conditional homoscedastic and nonautocorrelated error random variables

$E\left[\varepsilon_i^2 / f\left(X_j, \beta\right), j = 1, 2, \ldots, n\right] = \sigma^2$, a finite constant and

$E\left[\varepsilon_i \varepsilon_j / f\left(X_i, \beta\right), f\left(X_j, \beta\right) j = 1, 2, \ldots, n\right] = 0, \quad \forall j \neq i$

(v) The data generating process for $X_i$ is strictly exogenous to that of $\varepsilon_i$. The data on $X_i$ are assumed to be well behaved population such that the first and second order moments of the data can be assumed to converge to fixed, finite population counterparts.

(vi) $\varepsilon_i$'s are normally distributed.

i.e., $\varepsilon_i \overset{\text{i.i.d}}{\sim} N\left(O, \sigma^2\right), \qquad i = 1, 2, \ldots, n$

The nonlinear least squares (NLLS) estimator for $\beta$ is denoted by $\hat{\beta}$, which is defined as the value of $\beta$ that minimizes the residual sum of squares.

$$R\left(\hat{\beta}\right) = \sum_{i=1}^{n} \left[Y_i - f\left(X_i, \hat{\beta}\right)\right]^2$$

The first order condition for the minimization gives the nonlinear normal equations as

$$\frac{\partial R\left(\hat{\beta}\right)}{\partial \hat{\beta}} = \sum_{i=1}^{n} \left[Y_i - f\left(X_i, \hat{\beta}\right)\right] \frac{\partial f\left(X_i, \hat{\beta}\right)}{\partial \hat{\beta}} = 0$$

In general, there is no explicit solution for the NLLS estimator $\hat{\beta}$ and it can be obtained by some iterative method.

An estimate of the error variance corresponding to the NLLS estimator $\hat{\beta}$ is given by

## III. PROPERTIES OF NON LINEAR LEAST SQUARES ESTIMATOR

Suppose that the standard nonlinear regression model

$Y_i = f\left(X_i, \beta\right) + \varepsilon_i$, i=1,2,….,n may be written in matrix notation as,

$$Y_{nx1} = f_{nx1}\left(\beta\right) + \varepsilon_{nx1} \qquad \text{…(3.1)}$$

where,

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}_{n\times1}, f\left(\beta\right) = \begin{bmatrix} f\left(X_1, \beta\right) \\ f\left(X_2, \beta\right) \\ \vdots \\ f\left(X_n, \beta\right) \end{bmatrix}_{n\times1}$$

and

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}_{n\times 1}$$ are (nx1) vectors.

denote $F(\beta) = \left(\left(\frac{\partial}{\partial \beta_j} f(X_i, \beta)\right)\right)_{n\times p}$ ...(3.2)

where $\frac{\partial}{\partial \beta_j} f(X_i, \beta)$ is the (i, j)$^{th}$ element of matrix $F(\beta)$.

Without loss of generality, one may write F for $F(\beta)$ .

Now, the NLLS estimator of the unknown parametric vector $\beta$ is the nonlinear regression model $Y = f(\beta) + \varepsilon$ is the (px1) vector $\hat{\beta}$ which minimizes the residual sum of squares

$$R(\hat{\beta}) = \left[Y - f(\hat{\beta})\right]' \left[Y - f(\hat{\beta})\right] \qquad ...(3.3)$$

In large samples, the NLLS estimator is given by approximation

$$\hat{\beta} \approx \beta + (F'F)^{-1} F'\varepsilon \qquad ...(3.4)$$

Further, an estimate of error variance corresponding to the NLLS estimator is given by the approximation,

$$\hat{\sigma}^2 \approx \frac{\varepsilon' M \varepsilon}{n-p} \qquad ...(3.5)$$

where $M = \left[I - F(F'F)^{-1} F'\right]$ is a symmetric idempotent matrix such that $M'M = MM' = I$ .

Thus,

$\hat{\beta} \overset{asy}{\sim}$ Multivariate $N_p\left(\beta, \sigma^2 (F'F)^{-1}\right)$

and $\left[\frac{(n-p)\hat{\sigma}^2}{\sigma^2}\right]$ is independently $\overset{asy}{\sim} \chi^2_{(n-p)}$

Generally $(F'F)^{-1}$ may be approximated by $\Sigma^{-1} = \left[F'(\hat{\beta}) F(\hat{\beta})\right]^{-1}$ . Under certain regularity conditions, one may obtain

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \overset{asy}{\sim} N_p\left(0, \sigma^2 \Sigma^{-1}\right)$$

## IV. ITERATIVE PROCEDURE TO COMPUTE NONLINEAR LEAST SQUARES (NLLS) ESTIMATOR

Consider the least squares residual sum of squares for the minimization under nonlinear least squares estimation as

$$R\left(\hat{\beta}\right)=\sum_{i=1}^{n}\left[Y_i-f\left(X_i,\hat{\beta}\right)\right]^2 \qquad ...(4.1)$$

or
$$R\left(\hat{\beta}\right)=\left[Y-f\left(\hat{\beta}\right)\right]'\left[Y-f\left(\hat{\beta}\right)\right] \qquad ...(4.2)$$

By using the generalized Newton's Method for Numerical Analysis based on the substitution of the first order Taylor series expansion of Y around the initial estimate $\hat{\beta}_0$ in $R\left(\hat{\beta}\right)$, one may obtain,

$$f\left(\beta\right)\approx\lambda f\left(\hat{\beta}_0\right)+\left.\frac{\partial f}{\partial\beta'}\right|_{\hat{\beta}_0}\left(\beta-\hat{\beta}_0\right) \qquad ...(4.3)$$

And
$$R\left(\hat{\beta}\right)\approx\left[Y-\lambda f\left(\hat{\beta}_0\right)-\left.\frac{\partial f}{\partial\beta'}\right|_{\hat{\beta}_0}\left(\beta-\hat{\beta}_0\right)\right]'\left[Y-\lambda f\left(\hat{\beta}_0\right)-\left.\frac{\partial f}{\partial\beta'}\right|_{\hat{\beta}_0}\left(\beta-\hat{\beta}_0\right)\right] \qquad ...(4.4)$$

or $R\left(\hat{\beta}\right)\approx\left[Y-\lambda f\left(\hat{\beta}_0\right)-F\left(\hat{\beta}_0\right)\left(\beta-\hat{\beta}_0\right)\right]'\left[Y-\lambda f\left(\hat{\beta}_0\right)-F\left(\hat{\beta}_0\right)\left(\beta-\hat{\beta}_0\right)\right]$ ...(4.5)

where $F\left(\hat{\beta}_0\right)=\left.\frac{\partial f}{\partial\beta'}\right|_{\hat{\beta}_0}$ and is the multiplicity of the root in the generalized Newton's method for numerical analysis.

Minimization of approximating residual sum of squares $R\left(\hat{\beta}\right)$ is the R.H.S. of (4.5) with respect to $\beta$ at $\hat{\beta}_0$

Remark: If is a root of response function f(x) = 0 with multiplicity $\lambda$, then the Generalized Newton's Iteration formula for numerical analysis is given by

$$x_{n+1}=x_n-\lambda\frac{f\left(x_n\right)}{f'\left(x_n\right)} \qquad ...(4.6)$$

gives the second round estimator $\hat{\beta}_1$ of the iteration process as

$$\hat{\beta}_1=\hat{\beta}_0+\left[F'\left(\hat{\beta}_0\right)F\left(\hat{\beta}_0\right)\right]^{-1}F'\left(\hat{\beta}_0\right)\left[Y-\lambda f\left(\hat{\beta}_0\right)\right] \qquad ...(4.7)$$

This iteration process is to be continued until convergence is obtained.

### The algorithm for the iterative procedure is given by:

1. Choose an estimate $\hat{\beta}_0$ for $\beta$ ; an arbitrary value (1 or 2 or 3 ..................) for multiplicity $\lambda$ and compute

Q0 =
$$\left[F'\left(\hat{\beta}_0\right)F\left(\hat{\beta}_0\right)\right]^{-1}F'\left(\hat{\beta}_0\right)\left[Y-\lambda f\left(\hat{\beta}_0\right)\right] \qquad ...(4.8)$$

Find a value wo between 0 and 1 such that

$$R\left(\hat{\beta}_0+w_0Q_0\right)\leq R\left(\hat{\beta}_0\right) \qquad ...(4.9)$$

2. Let $\hat{\beta}_1 = \hat{\beta}_0 + w_0 Q_0$ . Compute

$$Q_1 = \left[F'\left(\hat{\beta}_1\right)F\left(\hat{\beta}_1\right)\right]^{-1}\left[Y - \lambda f\left(\hat{\beta}_1\right)\right] \qquad ...(4.10)$$

Find a value $w_1$ between 0 and 1 such that

$$R\left(\hat{\beta}_1 + w_1 Q_1\right) \le R\left(\hat{\beta}_1\right) \qquad ...(4.11)$$

3. Let $\hat{\beta}_2 = \hat{\beta}_1 + w_1 Q_1$

. . . .

. . . .

. . . .

The value for the step length wi at each iteration may be chosen by several methods.

The choice of initial values is an adhoc process. By the prior knowledge of the situation or data inspection or grid search or trial and error, the initial values may be chosen.

Alternatively, by inserting the approximation (4.3) at $\beta_1$ into the original nonlinear regression model (3.1) yields,

$$Y - \lambda f\left(\hat{\beta}_0\right) + F\left(\hat{\beta}_0\right)\hat{\beta}_0 \,\square\, F\left(\hat{\beta}_0\right)\beta_1 + \varepsilon \qquad ...(4.12)$$

or
$$Y^* = X^*\beta_1 + \varepsilon^* \qquad ...(4.13)$$

where $Y^* = Y - \lambda f\left(\hat{\beta}_0\right) + F\left(\hat{\beta}_0\right)\hat{\beta}_0$

and $X^* = F\left(\hat{\beta}_0\right)$

Here, $\varepsilon^*$ contains both the error and error in the first order Taylor series approximation to the true nonlinear regression. The least squares estimator of $\beta_1$ in the second round of iterative process say $\hat{\beta}_1$ can be obtained as

$$\hat{\beta}_1 = \left(X^{*\prime}X^*\right)^{-1}X^{*\prime}Y^* \text{ , where } X^* = F'\left(\hat{\beta}_0\right)$$

or
$$\hat{\beta}_1 = \left[F'\left(\hat{\beta}_0\right)F\left(\hat{\beta}_0\right)\right]^{-1}F'\left(\hat{\beta}_0\right)\left[Y - \lambda f\left(\hat{\beta}_0\right) + F\left(\hat{\beta}_0\right)\hat{\beta}_0\right]$$

or
$$\hat{\beta}_1 = \hat{\beta}_0 + \left[F'\left(\hat{\beta}_0\right)F\left(\hat{\beta}_0\right)\right]^{-1}F'\left(\hat{\beta}_0\right)\left[Y - \lambda f\left(\hat{\beta}_0\right)\right] \qquad ...(4.14)$$

Thus, by treating $F\left(\hat{\beta}\right)$ or $\left[\frac{\partial f}{\partial \beta'}\right]_{\hat{\beta}_0}$ as the regressor matrix, the nonlinear regression model reduces to a linear regression model.

Now, the solution to the minimization problem under iterator process is given by

$$\hat{\beta}_{n+1} = \hat{\beta}_n + \left[F'\left(\hat{\beta}_n\right)F\left(\hat{\beta}_n\right)\right]^{-1}F'\left(\hat{\beta}_n\right)\left[Y - \lambda f\left(\hat{\beta}_n\right)\right] \qquad ...(4.15)$$

Here, all the terms on the R.H.S of (4.15) are evaluated at $\hat{\beta}_n$ and $\left[Y - \lambda f\left(\hat{\beta}_n\right)\right]$ is the vector of nonlinear least squares residuals for an arbitrary value of $\lambda$ .

Thus, under iterative process, for each iteration, one may regress the nonlinear least squares residuals on the derivatives of the nonlinear regression functions and update (as zero) the previous estimates of the parameters. That is the iterative estimates will be consistent when $F'\left(\hat{\beta}_n\right)e_n$ is close enough to zero. Here $e_n = \left[Y - \lambda f\left(\hat{\beta}_n\right)\right]$ is the vector of nonlinear least squares residuals.

## V. CONCLUSIONS

Nonlinear model building has become an increasing important powerful tool in the subject of applied mathematics and Statistics. In recent years, the popularity of applications of nonlinear models has dramatically been rising up. Several researchers in applied mathematics are very often interested in inferential aspects of the nonlinear regression models. The nonlinear inferential methods and the error assumptions are generally analogous to those made for the linear regression models. The various mathematical methods in the numerical analysis can be applied to study the inferential aspects of estimators for the parameters in the nonlinear regression models. Some of the inferential questions with regard to the nonlinear models are still unanswered and offered a good research opportunity for the theoretical mathematicians and statisticians.

In the present study, a new iterative method of estimation of parameters of nonlinear regression model has been developed to compute Non Linear Least Squares (NLLS) estimator. Here NLLS residuals are regressed on the derivatives of the nonlinear regression functions in the iterative procedure. In the iterative process, the nonlinear studentized residuals can be regressed instead of least squares residuals on the derivatives of the nonlinear regression functions and update the previous parameter estimates.

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