Indian Streams Research Journal

QUASI-IDEALS -IN GAMMA-SEMI-NEAR-RINGS

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ABSTRACT: In this paper, the concept of quasi ideal in a Γ-semi near ring is introduced and investigated some of its properties.

AMS Classification: 08A72, 03E72.

KEY-WORDS: Γ-semi-nearring , quasi ideals of a Γ-semi-nearring.

§1. INTRODUCTION.

The concept of Γ- nearring was introduced by Satyanarayana [8]. Also M. K. Rao [6] studied Γ-semir ring and then N. K. Saha at el [7] defined the generalization of Γ-semirring and Γ- near ring as a Γ-semi-near rings and studied its properties. The notion of quasi-ideal, which is generalization of a left and a right ideal, is first introduced for semigroups by Stenfeld[11] and then for rings[10-12] and its properties. Iseki [5] introduced the concept of quasi-ideals for a semiring without zero and gave some characterization of it. Shabir et al [9] studied quasi-ideals of discussed quasi-ideals of Γ-semir ring

Chinram[2-3] studied quasi-ideals of Γ-semigroups and Γ-semir ring. Also Dutta[4] discussed quasi-ideals of Γ-semir ring In this Paper, we introduce the concept of quasi-ideals of a Γ- semi near ring and study its properties. Throughout this paper M denotes a right Γ-seminearring and we shall call it Γ-seminearring only unless otherwise specified.

§2. PRELIMINARIES.

We begin with the following definition.

Definition 2.1. Seminearring:-A nonempty set N together with two binary operations '+' and '·' satisfying the following conditions, is said to be a seminear ring.

(N, +) is a semigroup,

(N, ·) is a semigroup,

$(x+y)\cdot z = x\cdot z + y\cdot z$ for all x, y, z ∈ N.

Precisely speaking 'seminear ring' is a 'right seminear ring' here since every seminear ring satisfy one distributive law (left / right distributive law).

Every near ring is a seminear ring but every seminear ring need not be a nearring. For this we consider the following Example.

Example 1:-Let N = $\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ / a, b be nonnegative integers}, (N, +, .) is seminear ring under the matrix addition and matrix multiplication. Here N is a seminear ring which is not a near ring since (N, +) is a semigroup but not a group since additive inverse does not exist for all members of N.

Definition 2.2. Γ-near ring:-Let (M, +) be a group (need not be abelian) and Γ be a nonempty set. Then M= (M, +, Γ) is a Γ-near ring if there exists a mapping M× Γ ×M→M (the image of (x, α, y)→xαy) satisfying the following conditions :

i) (M, +, α) is a right near ring,

ii) xα (yβz) = (xαy) βz for all x, y, z ∈ M and α, β ∈ Γ.

Precisely speaking 'Γ-near ring' is a 'Γ-near ring'. Every near ring is a special type of Γ-near ring for singleton set Γ whereas every Γ-near ring a near ring for each member of Γ. See the following example.

Example 2: Let G = Z_8 = {0, 1, 2, ..7}, the additive group of integers modulo 8 and X= {a, b}. Define m_i: X→G, $m_i(a) = 0$, $m_i(b) = i$, for 0≤ i ≤ 7. such that M ={m_0 ,m_1 ,..., m_7} and let Γ = { g_0 ,g_1} where g_i :G→X define by

$g_0 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a & a & a & a & a & a & a & a \end{pmatrix}$,

$g_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a & a & a & b & a & a & a & b \end{pmatrix}$

For m ϵ M, g ∈ Γ, x ∈G. Take mgx = m(g(x)).
Then (M, +, Γ) becomes Γ-near-ring.

Definition 2.3. Γ-seminear ring:-Let M be an additive semigroup and Γ be a nonempty set. Then a semigroup (M, α) is called a right Γ-seminear ring if there exists a mapping M × Γ× M → M (denoted by (a, α, b) → aαb) satisfying the conditions:

i) (a+b) αc = aαc+bαc,

ii) aα(bβc) = (aαb)βc

for all a, b, c ∈ M and α, β ∈ Γ. Precisely speaking 'Γ-seminear ring' to mean 'right Γ-seminear ring'.

Every Γ-near ring is a Γ-seminear ring but every Γ-seminear ring need not be a Γ-near ring. For this we consider the following Example.

Example 3: Let M = {$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ / a, b be nonnegative integers} = Γ, Then (M, +, Γ) is Γ- seminear ring under the matrix addition and matrix multiplication. Here N is a seminear ring which is not a near ring since (M, +) is a semigroup but not a group since additive inverse does not exist for all members of M. Define M × Γ× M → M (denoted by (a, α, b) → aαb) where aαb is matrix multiplication of a, α, b Then M is a Γ-seminear ring but not a Γ-near ring. Since (M, +) is a semigroup which is not a group.

Definition 2.4. Sub-Γ-seminear ring:- Let M be a Γ-seminear ring. A nonempty subset M′ of M is a sub-Γ-seminear ring of M if M′ is also a Γ-seminear ring with the same operations of M.

Definition 2.5. Ideal of a Γ-seminear ring:-A subset I of a Γ-seminear ring M is a left (resp. right) ideal of a Γ-seminear ring M if I is a subsemigroup of M and rαx ∈I(resp. xαr ∈I) for all x, y ∈ I and r ∈ M, α ∈ Γ.

If I is both left as well as right ideal then we say that I is an ideal of M.

Example 4: Consider the example of Γ-seminear ring (M, +, .) mentioned above. We have I={$\begin{bmatrix} 2a & 2b \\ 0 & 0 \end{bmatrix}$ / a, b be nonnegative integers} is an ideal of M.

Definition 2.6 : Let I and J be two nonempty subsets of M, then we define

I + J = { i + j / i∈ I, j∈ J} and

I Γ J = { $\sum_{i=1}^{n} x_i\, \alpha_i\, y_i$ / x_i∈ I, α_i∈ Γ, y_i ∈ J}

If J is the set of natural numbers, then

J Γ = { $\sum_{i=1}^{n} x_i\, \alpha_i$ / x_i∈ J, α_i∈ Γ}.

Definition 2.7. Let X be two nonempty subset of M. By $(X)_l$ we mean the left ideal of M generated by X(i.e., intersection of all left ideal of M containing X).

Similarly $(X)_r$, $(X)_t$ denote the right and two sided ideal generated by X respectively.

Definition 2.8. Let left (right, two-sided) ideal I of a of Γ-seminear ring M is said to be left (right, two-sided) k-ideal of M if a, a+x ∈ I, then x ∈ I, for any x∈ M

§3. QUASI IDEALS OF A Γ-SEMINEAR RING.

Firstly we see the two basic results

Theorem 3.1. For each nonempty subset X of M the following two statements hold.

(i) M Γ X is a left ideal,

(ii) X ΓM is a right ideal,

(iii) . M Γ X Γ M is an ideal of M.

Proof. (i)

M Γ X ={ $\sum_{i=1}^{n} m_i\, \alpha_i\, x_i$ / m_i∈M, α_i∈ Γ, x_i ∈ X }

Let a, b ∈ M Γ X.

Then a+b = $\sum_{i=1}^{n} m_i\, \alpha_i\, x_i + \sum_{j=1}^{n} m_j\, \beta_j\, y_j$

Implies a+b is a finite sum. Hence a+b ∈ M Γ X and this shows M Γ X is a subsemigroup of (M, +). For t∈ M, a∈ MΓX, and β∈ Γ, then

$t\beta a = t\beta \sum_{i=1}^{n} m_i\, \alpha_i\, x_i) = \sum_{i=1}^{n} t\beta\, (m_i\, \alpha_i\, x_i) = \sum_{i=1}^{n} (t\beta m_i)\, \alpha_i\, x_i$ ∈ M Γ X.

Therefore M Γ X is a left ideal of M.

(ii) Similarly we can prove that XΓ M is a right ideal of M.
(iii)By (i)MΓX is a left ideal of M. Hence MΓXΓM is a right ideal of M by (ii).
(iv)Similarly, by (ii) XΓM is a right ideal of M. Hence MΓXΓM is a left t ideal of M by (i). Therefore MΓXΓM is an ideal of M.

Theorem 3.2. For any nonempty subset X of M we have

(i) If M has right unit element I, then $(X)_l$ = M Γ X,
(ii) If M has right unit element I, then $(X)_r$ = XΓM,,
(iii) If M has right unit element I, then $(X)_t$ = M ΓXΓM.

Proof. Let M contain left unit element I. Then I αa = a, for every a∈ M and α ∈ Γ. For any x ∈ X, x = 1αx ∈ MΓX. Hence X is a subset of MΓX. As MΓX is a left ideal of M, NX ⊆M Γ X. But then we have $(X)_l$ = NX +MΓX (see [2]). This iplies $(X)_l$ = MΓX +MΓX ⊆ M Γ X. As $(X)_l$ is the smallest left ideal of M containing X. This shows that $(X)_l$ = M Γ X. Similarly, we can prove that $(X)_r$ = XΓM and $(X)_t$ = M ΓXΓM.

Chinram [2] has defined a quasi ideal Q in a Γ-seminear ring M as follows.

Definition 3.3. A subsemigroup Q of (M, +) is a quasi-ideal of M if
(M ΓQ)∩ (Q ΓM) ⊆ Q.

Example 5. Let N be the set of natural numbers and Γ = 2N. Then N is a Γ-seminear ring and A = 3N is a quasiideal of a Γ-seminear ring N.

Example 6. Consider a Γ-seminear ring M = $M_2 \times_2 (N_0)$, where N_0 denotes the set of natural numbers with zero and Γ =M. Define A αB = usual matrix product of A, α and B ∈ M. Then Q = $\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ / a ∈N_0 } is a quasi-ideal of a Γ-seminear ring M.

Properties.

(1) By a quasi-ideal Q in a Γ-seminearring M we mean an additive subsemigroup of M such that (MΓQ)∩(Q ΓM) ⊆Q(See Iseki [5]). As every seminearring is a Γ-seminear ring. The two definitions given in [2] and [5] of quasi ideals coincide in a seminearring.

(2) Every quasi-ideal of M is a sub Γ-seminearring of M.

(3) **Every one sided ideal or two sided ideal of M is a quasi-ideal of M but converse need not be true.** For this consider Γ-seminear ring given in Example (5). Here Q = $\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ / a ∈N_0 } is a quasi-ideal but neither a left ideal nor a right ideal of M.

(4) If Q_1 and Q_2 are quasi-ideals of M, then $Q_1 \Gamma Q_2$ need not be a quasi-ideal of M. For this consider the following example

Example 7. If T = $\{\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$ / a, b ∈ R^+ }, then T is a semigroup with respect to usual matrix multiplication.

If M = $\{\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$ / a, b ∈ R^+ }∪ $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$ and Γ = M, then M is a Γ-seminearring with usual matrix multiplication. Define + in M by A + B = 0 if A, B ∈ M and A + 0 = 0 + A = A, for all A∈ M. If

Q_1 = $\{\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$ / a, b ∈ R^+ , 0 < a < b }∪ $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$ and

Q_2 = $\{\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$ / a, b ∈ R^+ , a> 0, b>5 }∪ $\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$.

Then Q_1 is a right ideal and Q_2 is a left ideals of M and hence Then Q_1 and Q_2 are quasi-ideals of M. But Q_1 Γ Q_2 is not quasi-ideals of M.

(5) **The sum of two quasi-ideals of M need not be a quasi-ideal of M.** We illustrate this by the following example.

Example 8. Let M = $M_2 \times_2 (N_0)$ be a Γ-seminear ring, If Γ = M, then M forms a Γ –seminearring with A αB = usual matrix product of A, α and B ∈ M for all A, α and B ∈ M.

Q_1 = $\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ / a ∈N_0 } and Q_2 = $\{\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ / a ∈N_0} are quasi-ideals of a Γ-seminear ring M. But Q_1 + Q_2 = $\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ / a , b ∈N_0} is not a quasi-ideal of M.

(6) Arbitrary intersection of quasi-ideals of M is either empty or a quasi-ideal of M.

Proof. Let $T = \bigcap_{i\in\Delta}\{Q_i / Q_i \text{ is a quasiideal of } M\}$, where Δ denotes any indexing set, be a nonempty set. T is a subsemigroup of (M, +). Further

$(M\Gamma T)\cap(T\Gamma M) = (M\Gamma(\bigcap_{i\in\Delta} Q_i))\cap((\bigcap_{i\in\Delta} Q_i)\Gamma M) \subseteq (Q_i\Gamma M)\cap(M\Gamma Q_i)$

$\subseteq Q_i$.

For all $i\in\Delta$. $(M\Gamma T)\cap(T\Gamma M) \subseteq \bigcap_{i\in\Delta} Q_i = T$. This shows that T is a quasi-ideal of M.

(7) The set of all quasi-ideals of M forms a Moore family and hence a complete lattice(see Birkhoff[1]).

(8) If Q is a quasi-ideal of M, then $Q^2 = Q\Gamma Q \subseteq Q$.

Proof. As Q is a quasi-ideal of M,$(M\Gamma X)\cap(X\Gamma M) \subseteq Q$. We have $Q^2 = Q\Gamma Q \subseteq Q\Gamma M$. Hence $Q^2 = Q\Gamma Q \subseteq (M\Gamma Q)\cap(X\Gamma M) \subseteq Q$. Thus $Q^2 = Q\Gamma Q = Q$.

(9) For each nonempty subset X of M, $(M\Gamma Q)\cap(X\Gamma M)$ is a quasi-ideal of M.

Proof. $M\Gamma(M\Gamma X)\cap(X\Gamma M)\Gamma M = (M\Gamma M)\Gamma X)\cap X\Gamma(M\Gamma M)$

$\subseteq (M\Gamma X)\cap(X\Gamma M)$.

Therefore, $(M\Gamma X)\cap(X\Gamma M)$ is a quasi-ideal of M.

(10) If M has an identity element 1, then every quasi-ideal of M is expressed as an intersection of a left ideal and a right ideal of M.

Proof. Let M be a Γ-seminear ring with an identity element 1. Let Q be a quasi-ideal of M. Then $(M\Gamma Q)$ is a left ideal and $(Q\Gamma M)$ is a right ideal of M(see result 3.1). As M contain an identity element 1, by result (3.2) we have $(Q)_l = M\Gamma Q$ and $(Q)_{lr} = Q\Gamma M$. Therefore $Q \subseteq (Q)_l = M\Gamma Q$ and $Q \subseteq (Q)_r = Q\Gamma M$ imply $Q \subseteq (M\Gamma Q)\cap(Q\Gamma M)$. But Q being a quasi-ideal of M, $(M\Gamma Q)\cap(Q\Gamma M) \subseteq Q$. Therefore $Q = (M\Gamma Q)\cap(Q\Gamma M)$. Thus every quasi-ideal of M is an intersection of a left ideal and a right ideal of M.

(11) Intersection of a right ideal and a left ideal of M is a quasi-ideal of M.

Proof. Let R be a right ideal and L be a left ideal ofM. Then $R\cap L)$ is a subsemigroup of (M, +).

Further $M\Gamma(R\cap L)\cap((R\cap L)\Gamma M) \subseteq (M\Gamma L)\cap(R\Gamma M) \subseteq L\cap R$. Hence $R\cap L$ is a quasi-ideal of M.

Recall that an element e of M is an idempotent elements in M. We obtain quasi-ideals in M. Now we prove the following.

Theroem 3.4. Let L be a left ideal of M. Then for any idempotent elements e of M, $e\Gamma L$ is a quasi-ideal of M.

Proof. First we prove that $e\Gamma L = L\cap(e\Gamma M)$. We know $(e\Gamma M)+(e\Gamma M) = e\Gamma(M+M) \subseteq e\Gamma M$. Hence $e\Gamma M$ is a subsemigroup of (M, +). As $(e\Gamma M)\Gamma M = e\Gamma(M\Gamma M) \subseteq e\Gamma M$, $e\Gamma M$ is a right ideal of M. As $e \in M$ and L is left ideal of M, $e\Gamma L \subseteq L$. Further $e\Gamma L \subseteq e\Gamma M$. These will imply $e\Gamma L \subseteq L\cap(e\Gamma M)$. For the reverse inclusion let $a \in L\cap(e\Gamma M)$..

Then $a = \sum_{i=1}^{n} e\,\alpha_i\,x_i$ for $xx_i \in M$, $\alpha_i \in \Gamma$.

Thus $a = \sum_{i=1}^{n} e^2\,\alpha_i\,x_i = \sum_{i=1}^{n}(e\alpha e)\,\alpha_i\,x_i = e\alpha\sum_{i=1}^{n} e\,\alpha_i\,x_i = e\alpha a \in e\Gamma L$. This shows that $L\cap(e\Gamma M) \subseteq e\Gamma L$,.

Hence $L\cap(e\Gamma M) = e\Gamma L$.

As L is a left ideal and $e\Gamma M$ is a right ideal of M we get $e\Gamma L$ is a quasi-ideal of M(see Property (11)).

As in Theorem 3.1.3 we can prove the following therorem.

Theorem 3.5. Let R be a right ideal. Then for any idempotent elements e of M. $R\Gamma e$ is a quasi-ideal of M.

Theorem 3.6. Let R be right ideal and L be a left ideal of M. Then for any idempotent elements e, f of M, $e\Gamma M\Gamma f$ is a quasi-ideal of M.

Proof. First we prove that , $e\Gamma M\Gamma f = (e\Gamma M)\cap(M\Gamma e)$.

$e\Gamma M\Gamma f = (e\Gamma M)\Gamma f \subseteq e\Gamma M$ and $e\Gamma M\Gamma f = e\Gamma(M\Gamma f) \subseteq M\Gamma e$. Thus

$e\Gamma M\Gamma f \subseteq e\Gamma M\cap(M\Gamma e)$. Let $a \in (M\Gamma e)\cap(e\Gamma M)$

Then $a = \sum_{i=1}^{n} x_i\ \alpha_i f = \sum_{i=1}^{n} x_i\ \alpha_i(f\,\alpha f) = \sum_{i=1}^{n}(x_i\,\alpha_i\,f)\,\alpha f = \sum_{j=1}^{n}(e\,\beta_j\ y_j)\,\alpha f$

Thus $a = \alpha f$, for all $\alpha \in \Gamma$. As $a \in e\Gamma M$, $\alpha \in \Gamma \Longrightarrow a = a\alpha f \in e\Gamma M\Gamma f$.

We get $(e\Gamma M)\cap(M\Gamma e) \subseteq e\Gamma M\Gamma f$. Thus $(e\Gamma M)\cap(M\Gamma e) = e\Gamma M\Gamma f$. As $M\Gamma f$ is a left ideal and $e\Gamma M$ is a right ideal of M we get $(e\Gamma M)\cap(M\Gamma e) = e\Gamma M\Gamma f$ is a quasi-ideal of M(see property (11)).

Intersection of a quasi-ideal and a sub a Γ-seminear ring of M is a quasi-ideal of that a subΓ-seminear ring of M. We prove that in the following theorem.

Theorem 3.7. If Q is a quasi-ideal and T is a sub Γ-seminear ring of M the $Q\cap T$ is a quasi-ideal of T.

Proof. As $Q \cap T$ is a subsemigroup of $(M, +)$ and $Q \cap T$ is a subsemigroup of $(T, +)$. Further,
$T\Gamma(T \cap Q) \cap (T\Gamma Q)\Gamma T \subseteq (T\Gamma Q) \cap (Q\Gamma T) \subseteq (M\Gamma Q) \cap (Q\Gamma M) \subseteq Q$.
And $T\Gamma(T \cap Q) \cap (T \cap Q)\Gamma T \subseteq (T\Gamma T) \cap (T\Gamma T) \subseteq T \cap T \subseteq T$.
This implies $T\Gamma(T \cap Q) \cap (T \cap Q)\Gamma T \subseteq Q \cap T$. This shows that $Q \cap T$ is a quasi-ideal of T.

Definition 3.8. A Γ-seminear ring M is said to be a quasi-simple Γ-seminear ring if M is unique quasi-ideal of M, i.e., M has no proper quasi-ideal.

A characterization of quai-simple Γ-seminear ring is given in the following theorem.

Theorem 3.9. If M is a Γ-seminear ring, then M is quasi-simple Γ-seminear ring if and only if $(M\Gamma a) \cap (a\Gamma MT) = M$, for all $a \in M$.

Proof. Suppose M is a quasi-simple Γ-seminear ring. For any $a \in M$. $(M\Gamma a)$ and $(a\Gamma M)$ are left and right ideals of M respectively. Therefore $(M\Gamma a) \cap (a\Gamma M)$ is a quasi-ideal of M(see property(11)). Further $(M\Gamma a) \subseteq M$ and $(a\Gamma M) \subseteq M$ imply $(M\Gamma a) \cap (a\Gamma M) \subseteq M$. As M is a quasi-simple Γ-seminear ring , $M = (M\Gamma a) \cap (a\Gamma M)$.
Conversely, suppose that , $M = (M\Gamma a) \cap (a\Gamma M)$. Let Q be quasi-ideal of M. For any $q \in Q$, by assumption we have, $M = (M\Gamma q) \cap (q\Gamma M) \subseteq (M\Gamma Q) \cap (Q\Gamma M) \subseteq Q$. Therefore $M \subseteq Q$. Thus $M = Q$. Hence M is a quasi-simple Γ-seminear ring.

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