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A NOTE ON Γ –NEAR SUBTRACTION SEMIGROUPS

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ABSTRACT.

In this paper, we introduce generalization of near subtraction semigroup as a Γ-near subtraction semigroup and its ideals and study some of its properties.

Keywords. Γ-near subtraction semigroup, Ideals of Γ-near subtraction semigroup, insertion of factors property [IFP].

AMClassification. 06F35

1. INTRODUCTION

Schein B. M. [6] introduced the set theoretic subtraction '-' which is analogous to subtraction algebra and it is developed by Abbott J. C.[1]. Zelinka B. [7] discussed the problem of Schein relating the structure of multiplication in a subtraction semigroup. Kim K. H. at el[3] studied an ideal of a subtraction semigroup. Dheena P.and his colleagues[2] defined the concept of near - subtraction semigroups and discussed its properties. Pilz G. [3] defined near rings and Satynarayan Bh.[4] introduced Γ-nearring and studied its various properties. In this paper, we introduce the concept of Γ-near subtraction semigroup which is the generalization of near subtraction semigroup and discussed its properties.

2. PRELIMINARIES

We recalled the following definitions and its properties :

Definition 2.1. Let A be a nonempty set and subtraction '-' is a single binary operation. Then an algebra $(A, -)$ is said to be a subtraction algebra if it satisfies the following axioms:
For any $a, b, c \in A$,
(i) $a - (b - a) = a$;
(ii) $a - (a - b) = b - (b - a)$;
(iii) $(a - b) - c = (a - c) - b$.
In (iii) omition of parentheses in expressions of the form $(x - y) - z$ is allowed.
In the subtraction algebra, the following properties hold:
P1. $a - 0 = a$ and $0 - a = 0$.

P2. $a - (a - b) = 0$.
P3. $(a - b) - b = (a - b)$.
P4. $(a-b)- (b - a)= a – b$ where $a–a = 0$ where element a does not depend on the choice of $a\in A$.
P5. $a - (a - (a - b)) = a - b$.

Definition 2.2. A non empty subset S of a subtraction algebra N is said to be a subalgebra of N, if $a - b \in S$ whenever a, $b \in S$.
Definition 2.3. A subtraction semigroup is an algebra $(A, \cdot, -)$ with two binary operations '-'and '.'that satisfies the following properties:
For any $a, b, c \in A$,
1. (A, .) is a semigroup,
2. (A, -) is a subtraction algebra,
3. $a(b - c) = ab - ac$ and $(a - b)c = ac - bc$.
A subtraction semigroup is said to be multiplicatively abelian if multiplication is commutative.
Definition 2.4. A non-empty set N together with the binary operations "–" and "." is said to be a near-subtraction semigroup if it satisfies the following:
1. (N, -) is a subtraction algebra.
2. (N, .) is a semigroup.
3. $(a - b)c = ac - bc$,for all $a, b,c \in N$.
It is clear that $0a = 0$,for all $a \in N$. Similarly we can define a near-subtraction semigroup (left).
We always take a near-subtraction semigroup means it is a near-subtraction semigroup(right) only.

3. Γ-NEAR SUBTRACTION SEMIGROUP.

Definition 3.1. Let (N, -) be a near-subtraction semigroup and $\Gamma =\{ \alpha, \beta, ..\}$ be a nonempty set of operators. Then N is said to be a Γ-near subtraction semigroup, if there exists a mapping $N \times \Gamma \times N \to N$ (the image of (a, α, b) is denoted by $a\alpha b$), satisfying the following conditions:
1. (N, α) is a semigroup, $\alpha \in \Gamma$
2. (N, -) is a subtraction algebra,
3. $(a - b)\alpha c = a\alpha c – b\alpha c$ (right distributive law),
4. $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.
In practice we called simply 'Γ-near subtraction semigroup' instead of 'right Γ-near subtraction semigroup' . Similarly we can define a Γ-near subtraction semigroup (left). It is clear that $0\alpha a = 0$,for all $a \in N$ and $\alpha \in \Gamma$.

Example 3.2. Let N = {0, 1, 2, 3, 4, 5} in which '-' and $\alpha \in \Gamma$ are defined by

-	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

α	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	5
5	0	0	5	0	0	5

β	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	5
5	0	0	5	0	0	5

It is easily verified that N is a Γ-near subtraction semigroup.
In this paper, we denote the set of all non-zero elements of N i.e., $N* = N - \{0\}$ and T denotes the set of all idempotent elements of T ($t \in T$ if and only if $t^2 =t\alpha t=t$) and V denotes the set of all nilpotent elements of N ($a \in V$ if and only if $a^k = a\alpha a\alpha a\alpha \ldots$ k times= 0 for some positive integer k).An ideal I of N is said to be nil if every element of N is nilpotent. Further N is called a nil Γ- near subtraction semigroup if every element of N is nilpotent.
In a right Γ- near subtraction semigroup N, $0\alpha a = 0$ for all $a \in N$. But $a\alpha 0$ need not be equal to 0, for $a \in N$. So there is a need to define the following:

Definition 3.3. (i) The set $\{a \in N/a\alpha 0 = 0\}$ is called the zero-symmetric part of N and is denoted by N0.
(ii) A right Γ- near subtraction semigroup N is said to be zero symmetric if N = N0.
Example 3.2. verifies that $(N,-, \alpha)$ for $\alpha \in \Gamma$ is a zero symmetric right Γ- near subtraction semigroup i.e., N = N0.
Now we introduce the ideals of Γ-near subtraction semigroup.

Definition 3.4. Let $(N,-, \alpha)$ for $\alpha \in \Gamma$ be a Γ-near subtraction semigroup. A non empty subset I of N is called
(i) a left ideal if I is a subalgebra of (N,−) and $a\alpha i - a\alpha(a' - i) \in I$ for all $a, a' \in N$ and $i \in I$

(ii) a right ideal if I is a subalgebra of (N,−) and IΓN ⊆ I
(iii) an ideal if I is both a left ideal and a right ideal
Proposition 3.5. Let I ⊆ N. Then the following are equivalent.
(i) For all a ∈ I, b ∈ N, a − b ∈ I
(ii) a ≤ b and b ∈ I ⇒ a ∈ I.

Proof. (i) ⇒ (ii): Suppose a ≤ b and b ∈ I. Then a − b = 0. Now a = a – 0 = a − (a − b) = b − (b − a) ∈ I Therefore a ∈ I.
(ii) ⇒ (i): Let a ∈ I and b ∈ N. Now (a − b) − a = 0 ⇒ (a − b) ≤ a. Hence (a − b) ∈ I.

Proposition 3.6. (i) Suppose N is a zero symmetric Γ-near subtraction semigroup and I is a left ideal of N such that a –b for every a, b ∈ N Then the following are equivalent:
(i) NΓI ⊆ I
(ii) aαi−aα(a′−i) ∈ I for all a, a′ ∈ N, α ∈ Γ and i ∈ I.

Proof. (i)⇒(ii) Let a, a′ ∈ N, α ∈ Γ and i ∈ I. By (i) aαi∈ I. Since I is a left ideal of N we have for a ∈ I and a′ ∈ N, a- a′ ∈ N and aαi−aα(a′−i) ∈ I for all a, a′ ∈ N, α ∈ Γ and i ∈ I.

(ii)⇒(i)If N is a zero-symmetric Γ-near subtraction semigroup and I is a left ideal of N, then for i ∈ I and a ∈ N, we have aαi−aα(0−i) = aαi−0 = aαi ∈ I. That is NΓI ⊆ I.

Definition 3.7. If A,B ⊆ N , a Γ-near subtraction semigroup, then
(i) A − B = {a – b / a ∈ A, b ∈ B}
(ii) AB or AαB = {aαb / a ∈ A, b ∈ B}
(iii) If a ∈ N then Na or Nαa = {bαa/b ∈ N}. We say that a subset M of N which is closed under '−' and for α ∈ Γ, NΓM ⊂ M is an N-system. If, in addition, MΓN ⊂ M then M is called an invariant N-system. Obviously, for every a ∈ N, Na is an N-system and is also called a Principal N-system.

Proposition 3.8. (i) If N is a Γ-near subtraction semigroup then the concepts of left ideals and ideals of N coincide with N-systems of N and invariant N-systems respectively.
(ii) N is zero symmetric if and only if every left ideal of N is an N-system of N.
(But an N-system of N need not be a left ideal of N in general).
(iii) N_0 is a left ideal of N, but not necessarily an ideal of N.

Definition 3.9. Let N be a Γ-near subtraction semigroup. We define the following
1. N is said to be a Γ-near subtraction semigroup with identity if there exists an element 1 ∈ N, α ∈ Γ such that 1αa = aα1 = a for every a ∈ N.
2. An element u of N is said to be a unit if there exists an element v ∈ N, α ∈ Γ such that uαv = 1.
3. N is said to be left bipotent if Na = Na^2 for all a ∈ N.
4. N is said to be subcommutative if Na = aN for all a ∈ N.
5. A Γ-subtraction semigroup N is said to be Von-Neumann regular if for every a ∈ N, there exists b ∈ N, α ∈ Γ such that a = aαbαa.
6. A Γ-near subtraction semigroup N is said to be Boolean if and only if a^2=aαa= a for all a ∈ N.
7. An N-system A of N is called essential if A ∩ B = {0} whenever B is any N-system of N, then B = {0}.
8. We say that, the element u is a right (left) zero divisor, if there exists an element v ≠ 0 in N, α ∈ Γ such that vαu = 0 (uαv = 0).
9. N is said to have property P4 if for all ideals I of N and for a, b∈ N, α ∈ Γ, aαb∈ I ⇒ bαa∈ I.
10. Anonempty subset A of N is called a multiplicative system if A is closed under multiplication.
11. A Γ-near subtraction semigroup N is said to be simple if Na = N for all a ∈ N.

Definition 3.10. An ideal I of N is called
(i) a prime ideal if for all ideals A,B of N, AB ⊆ I ⇒ A ⊆ I or B ⊆ I.
(ii) a semiprime ideal if for all ideals I' of N, I'αI' =I'^2 ⊆ I ⇒ I' ⊆ I.
(iii) a completely semiprime ideal if for any a in N, aαa = a^2 ∈ I ⇒ a ∈ I.
(iv) a principal ideal if I = Na for some a ∈ N.
(v) a primary ideal if aαbαc ∈ I, α ∈ Γ and if the product of any two of a, b, c is not in I, then the k^{th} power of the third element is in I.
(vi) a strictly prime ideal if for any two N-systems A, B of N, AB ⊂ I ⇒ A ⊂ I or B ⊂ I.
(vii) N is called a strictly prime Γ-near subtraction semigroup if {0}is a strictly prime ideal.
(viii) N is subdirectly irreducible if and only if the intersection of all non-zero ideals of N is non zero.
Similar to Definition 3.10 we have the following

Definition 3.11. (i) An N-system M of a near subtraction semigroup N is said to be a prime N-system if AB or $A\alpha B \subset M \Rightarrow A \subset M$ or $B \subset M$, for all N-systems A, B of N.
(ii) An N-system M of N is said to be completely prime N-system if $a\alpha b \in M \Rightarrow a \in M$ or $b \in M$.
(iii) An N-system M of N is said to completely semiprime N-system if $a^2 = a\alpha a \in M \Rightarrow a \in M$.
(iv) An N-system M of N is said to be primary N-system if for $\alpha \in \Gamma$ $a\alpha b\alpha c \in M$ and if the product of any two of a, b, c is not in M, then the k^{th} power of the third element is in M.
(v) An N-system M of N is said to be maximal N-system if it is maximal in the set of all non-zero N-systems of N.

Now we discuss below the results for Γ- near subtraction semigroups which are similar to ring.

Proposition 3.12. N has no non-zero nilpotent elements if and only if $a^2 = a\alpha a = 0 \Rightarrow a = 0$ for all $a \in N$, $\alpha \in \Gamma$.

Definition 3.13. A Γ- near subtraction semigroup N is said to have **I**nsertion of **F**actors **P**roperty [IFP] for short - if for a, b in N, $a\alpha b = 0 \Rightarrow a\alpha c\alpha b = 0$ for all $c \in N$, $\alpha \in \Gamma$.

Proposition 3.14. If N is a zero symmetric Γ-near subtraction semigroup then the following assertions are equivalent.
(i) N has IFP
(ii) For each $a \in N$, (0, a) is an ideal of N
(iii) For each subset M of N, (0, M) is an ideal of N.

Proof. (i) ⇒ (ii): For $p_1, p_2 \in (0, a)$, $\alpha \in \Gamma$, $(p_1 - p_2)\alpha a = p_1\alpha a - p_2\alpha a = 0$. Therefore $p_1 - p_2 \in (0, a)$. Let $b, b' \in N$ and $i \in (0, a)$. Then $(b\alpha i - b\alpha(b' - i))\alpha a = b\alpha i\alpha a - b\alpha(b' - i)\alpha a = b\alpha 0 - b\alpha(b' - i)\alpha a = 0 - b\alpha(b' - i)\alpha a = 0$. And $i\alpha b\alpha a = 0$ (by IFP). Thus (0, a) is an ideal of N, for every $a \in N$.
In a similar fashion we can prove (ii) ⇒ (iii).
(iii) ⇒ (i): Let $a, b \in N$, $\alpha \in \Gamma$ such that $a\alpha b = 0$. Then $a \in (0, b)$. Hence by (iii) $a\alpha c \in (0, b)$ for every $c \in N$. Thus $a\alpha c\alpha b = 0$ for every $c \in N$.

Definition 3.15. We say that N has (∗, IFP) if, (i) N has IFP and(ii) $a\alpha b = 0 \Rightarrow b\alpha a = 0$, for $a, b \in N$, $\alpha \in \Gamma$.

Proposition 3.16. Let N be zero symmetric Γ-near subtraction semigroup without non-zero nilpotent elements. Then N has (∗, IFP).

Proof. Suppose $a\alpha b = 0$ for some $a, b \in N$, $\alpha \in \Gamma$. Then $(b\alpha a)^2 = b\alpha(a\alpha b)\alpha a = b\alpha 0\alpha a = 0$ [since $N = N_0$]. Since N has no non-zero nilpotent elements, $b\alpha a = 0$. Also for any $n \in N$, $(a\alpha n\alpha b)^2 = (a\alpha n\alpha b)(a\alpha n\alpha b) = a\alpha n\alpha(b\alpha a)\alpha n\alpha b = a\alpha n\alpha 0\alpha n\alpha b = 0$. Consequently $a\alpha n\alpha b = 0$.Thus N has (∗, IFP).

Definition 3.17. For $a, b \in N$, we define an N-homomorphism as a map $f : Na \to Nb$ satisfying $f(c_1\alpha a - c_2\alpha a) = f(c_1\alpha a) - f(c_2\alpha a)$ and $f(n\alpha c\alpha a) = n\alpha(f(c\alpha a))$ for all $n \in N$, $\alpha \in \Gamma$.

Proposition 3.18. Let N be a Γ- near subtraction semigroup without non-zero nilpotent elements. If $a, b \in N$, $\alpha \in \Gamma$ and $t \in T$ (That is, t is an idempotent of N), then $a\alpha b\alpha t = a\alpha t\alpha b$.

Proof. Proposition 3.16. demands that N has (∗, IFP). Let t be an idempotent in N. For every $a, b \in N$, $\alpha \in \Gamma$ since $(a - a\alpha t)\alpha t = 0$, we have $(a - a\alpha t)\alpha b\alpha t = 0$ so that $a\alpha b\alpha t - a\alpha t\alpha b\alpha t = 0$. Also $(a\alpha t - a)\alpha t = 0 \Rightarrow (a\alpha t - a)\alpha b\alpha t = 0 \Rightarrow a\alpha t\alpha b\alpha t - a\alpha b\alpha t = 0$. Hence $a\alpha b\alpha t = a\alpha t\alpha b\alpha t$. Since $(t\alpha b - t\alpha b\alpha t)\alpha t = 0$, we get $t\alpha b\alpha(t\alpha b - t\alpha b\alpha t) = 0$ and $t\alpha b\alpha t\alpha(t\alpha b - t\alpha b\alpha t) = 0$. It follows that $(t\alpha b - t\alpha b\alpha t)^2 = 0$. Since N has no non-zero nilpotent element we get $(t\alpha b - t\alpha b\alpha t) = 0$. Hence $(t\alpha b\alpha t - t\alpha b) = 0$. Thus $t\alpha b = t\alpha b\alpha t$. Similarly $a\alpha b\alpha t = a\alpha t\alpha b$.

Definition 3.19. N has strong IFP if and only if for all ideals I of N and for all $a, b, n \in N$, $\alpha \in \Gamma$, $a\alpha b \in I \Rightarrow a\alpha n\alpha b \in I$.

Definition 3.20. Let N_1 and N_2 be two Γ- near subtraction semigroups. A map $f : N_1 \to N_2$ is said to be Γ-near subtraction semigroup homomorphism if
(i) $f(a - b) = f(a) - f(b)$
(ii) $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in N_1$, $\alpha \in \Gamma$.

Definition 3.21. A mapping f is said to be a Γ-isomorphism if f is one-one and onto.

Definition 3.22. The quotient Γ-near subtraction semigroups N/I is set of cosets of I where I is an ideal of a Γ-near-ring.

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